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Journal of Pure and Applied Algebra 193 (2004) 61–70
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Nilpotent symmetric Jacobian matrices and the Jacobian conjecture

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Received 28 June 2003; received in revised form 19 February 2004
Communicated by C.A. Weibel

Abstract

Let $H:\mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that the Jacobian $\mathcal{J}H$ of H is nilpotent and symmetric. The symmetric dependence problem, $SDP(n)$, asks whether the rows of the matrix $\mathcal{J}H$ are dependent over $\mathbb{C}$. We show that if $SDP(r)$ has an affirmative answer for all $r \leqslant n$, then the Jacobian conjecture holds for all $F:\mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$ with $\mathcal{J}H$ nilpotent and symmetric. As a consequence, we deduce the main result of (J. Pure Appl. Algebra, 189/1–3, 123–133), which asserts that the Jacobian conjecture holds for all polynomial maps of the form $F = x + H$, with $\mathcal{J}H$ nilpotent, symmetric and homogeneous, and $n \leqslant 4$.

MSC: 14R15; 14R10; 14E07

0. Introduction

Write $\mathcal{J}F$ for the Jacobian of a polynomial map $F:\mathbb{C}^n \to \mathbb{C}^n$. The Jacobian conjecture claims that F is an invertible polynomial map in case $\det \mathcal{J}F \in \mathbb{C}^*$. It was shown in [4] that in case $n \leqslant 4$, the Jacobian conjecture holds for all polynomial maps $F:\mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$, where H is homogeneous and $\mathcal{J}H$ is nilpotent and symmetric. Let $\mathcal{H}f$ be the matrix

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0022-4049/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.jpaa.2004.03.003

defined by

$$\mathscr{H} f = \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f & \frac{\partial^2}{\partial x_2 \partial x_1} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} f \\ \frac{\partial^2}{\partial x_1 \partial x_2} f & \frac{\partial^2}{\partial x_2 \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} f \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2}{\partial x_1 \partial x_n} f & \frac{\partial^2}{\partial x_2 \partial x_n} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_n} f \end{pmatrix}.$$

The main ingredient in the proof is a result due to Gordan and Nöther in [5], which asserts the following: if $n \leqslant 4$ and $h \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is a homogeneous polynomial such that $\det \mathscr{H} h = 0$, then h is degenerate, i.e. there exists a $T \in GL_n(\mathbb{C})$ such that $h(Tx) \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$.

In this paper we generalize the main result of [4] to the n-dimensional case. More precisely, we show that if F is of the form $F = x + H$ with $\mathscr{J} H$ nilpotent and symmetric (H does not need to be homogeneous), then F is invertible, provided a certain dependence problem ($SDP(n)$ in Section 1) has an affirmative answer. Since the Gordan–Nöther theorem implies that the homogeneous dependence problem ($HSDP(n)$ in Section 1) has an affirmative answer for $n \leqslant 4$ (Corollary 1.3) and $SDP(n)$ has an affirmative answer for $n \leqslant 2$ (Proposition 1.1), our main theorem (Theorem 2.1) implies the main result of [4].

The interest of studying the symmetric case comes from the fact that in [3], the authors have reduced the Jacobian conjecture to this case.

1. The symmetric dependence problem

Throughout this paper K denotes a field of characteristic zero and $K[x] = K[x_1, x_2, \ldots, x_n]$ is the polynomial ring in n indeterminates over K. In search of the Jacobian conjecture the following problem arose naturally (see [7, Conjecture 1, p. 80], [8, Conjecture B, p. 135], [9, Conjecture 11.3], [1] and [2, 7.1.7]).

Dependence problem $DP(n)$**.** Let $H = (H_1, H_2, \ldots, H_n) \in K[x]^n$ such that $\mathscr{J} H$ is nilpotent. Are the rows of $\mathscr{J} H$ dependent over K?

It is not difficult to see that, in case $H_i(0) = 0$ for all i, the dependence of the rows of $\mathscr{J} H$ is equivalent to the linear dependence of the polynomials $H_1, H_2, \ldots, H_n$ over K.

Due to the embedding lemma (Lefschetz principle) (see [2, Lemma 1.1.13]), we only need to examine the case $K = \mathbb{C}$ in the above and subsequent dependence problems.

In case $n \leqslant 2$, the dependence problem has an affirmative answer, however if $n \geqslant 3$ then there are counterexamples (see [2, Theorem 7.1.7]). On the other hand, if we additionally assume that each H_i is either zero or homogeneous of a fixed degree $d \geqslant 1$, then the corresponding problem is still open for all $n \geqslant 3$:

Homogeneous dependence problem *HDP*(n)**.** Let $H = (H_1, H_2, \dots, H_n) \in K[x]^n$ be homogeneous of degree $d \geqslant 1$ such that $\mathcal{J}H$ is nilpotent. Are the rows of $\mathcal{J}H$ dependent over K?

In fact a highly non-trivial result obtained by Hubbers in [6] (see also [2, Theorem 7.1.2]) completely classifies all such maps H in case $n=4$ and $d=3$. From this result it follows that the homogeneous dependence problem has an affirmative answer in this case, see [2, Corollary 7.1.4].

In this section we discuss the dependence problem for symmetric nilpotent Jacobian matrices. So let $F=(F_1, F_2, \dots, F_n) \in K[x]^n$ and assume that $\mathcal{J}F$ is a symmetric matrix. Then $F=(\mathcal{J}f)^{\mathrm{t}}$ for some $f \in K[x]$ (see for example [2, Lemma 1.3.53]). Consequently, a symmetric Jacobian matrix is of the form

$$\mathcal{H}f = \begin{pmatrix} \dfrac{\partial^2}{\partial x_1 \partial x_1} f & \dfrac{\partial^2}{\partial x_2 \partial x_1} f & \cdots & \dfrac{\partial^2}{\partial x_n \partial x_1} f \\ \dfrac{\partial^2}{\partial x_1 \partial x_2} f & \dfrac{\partial^2}{\partial x_2 \partial x_2} f & \cdots & \dfrac{\partial^2}{\partial x_n \partial x_2} f \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial^2}{\partial x_1 \partial x_n} f & \dfrac{\partial^2}{\partial x_2 \partial x_n} f & \cdots & \dfrac{\partial^2}{\partial x_n \partial x_n} f \end{pmatrix}.$$

The above matrix $\mathcal{H}f$ is called the Hessian of f. Observe that $\mathcal{H}f = \mathcal{H}\tilde{f}$, where $\tilde{f}$ is obtained from f by subtracting all monomials of degree $\leqslant 1$. So $\tilde{f}$ has only monomials of degree $\geqslant 2$. Such a polynomial is called *reduced*. Furthermore, we call $\tilde{f}$ the reduced part of f. Of course, the reduced part of a reduced polynomial is the polynomial itself.

Formulating the (homogeneous) dependence problem for symmetric Jacobian matrices then gives:

Symmetric dependence problem *SDP*(n)**.** Let $h \in K[x_1, x_2, \dots, x_n]$ such that $\mathcal{H}h$ is nilpotent. Are the rows of $\mathcal{H}h$ dependent over K?

Homogeneous symmetric dependence problem *HSDP*(n)**.** Let $h \in K[x_1, x_2, \dots, x_n]$ be homogeneous of degree $d \geqslant 2$ such that $\mathcal{H}h$ is nilpotent. Are the rows of $\mathcal{H}h$ dependent over K?

The following proposition is an immediate consequence of *DP*(2), but a direct proof is easier and gives some ideas about nilpotent Hessians.

Proposition 1.1. *SDP*(2) *has an affirmative answer.*

Proof. Let $M \in Mat_2(K[x])$ be symmetric and nilpotent (M does not need to be a Hessian matrix). Then $M^2 = 0$. So the first row v^t of M satisfies $v^t M = 0$. Since v is the first column of M, $v^t \cdot v = 0$. In other words, v is orthogonal to itself with respect to the common bilinear form $\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i$ (here $n = 2$). Such a vector is called *isotropic*.

Since v is isotropic, $v_2^2 = -v_1^2$, i.e. $v_2 = \pm \mathrm{i} v_1$, where $\mathrm{i} = \sqrt{-1}$. Consequently, $(1, \pm\mathrm{i}) \cdot M = 0$. In particular, $\mathrm{i} \in K$ or $M = 0$. □

One can easily see that for $j \geqslant r/2$ and symmetric $M \in Mat_n(K[x])$ with $M^r = 0$ and $M^{r-1} \neq 0$, all rows of M^j are orthogonal to each other. In particular, all rows of M^j are isotropic. Furthermore, all nonzero rows of M^{r-1} are eigenvectors of M. Isotropic vectors seem to play an important role in the theory of nilpotent Hessians.

Now we relate the $HSDP(n)$ to the Gordan–Nöther theorem mentioned in the introduction. Let $f \in K[x]$ and $T \in GL_n(K)$. Put $f \circ T = f(Tx)$. Then it is well-known that

$$\mathscr{H}(f \circ T) = T^t (\mathscr{H} f)|_{Tx} T, \tag{1}$$

where $M|_{Tx}$ is the matrix with entries $M_{ij}(Tx)$.

We call f *degenerate* if there exists a $T \in GL_n(K)$ such that $f \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$.

Proposition 1.2. *Let $f \in K[x_1, x_2, \ldots, x_n]$. Then the following statements are equivalent.*

(i) *The rows of $\mathscr{H} f$ are dependent over K.*
(ii) *The columns of $\mathscr{H} f$ are dependent over K.*
(iii) *There exists a nonzero $v \in K^n$ such that $\mathscr{H} f \cdot v = 0$.*
(iv) *There exists a $T \in GL_n(K)$ such that the last column of $\mathscr{H}(f \circ T)$ equals zero.*
(v) *The reduced part $\tilde{f}$ of f is degenerate, i.e. there exists a $T \in GL_n(K)$ such that $\tilde{f} \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$.*

Moreover, the $T \in GL_n(K)$ for which (iv) *holds match those for which* (v) *holds.*

Proof. (iii) is a reformulation of (ii). Further, (i) and (ii) are equivalent, since $\mathscr{H} f$ is symmetric. So it suffices to show (iii) ⇒ (iv), (iv) ⇒ (v), and (v) ⇒ (iii).

First assume (iii). Extend v to a $T \in GL_n(K)$ such that v is the last column of T. Then the last column of $\mathscr{H} f \cdot T$ equals $\mathscr{H} f \cdot v = 0$. From (1), (iv) now follows.

Next assume (iv). Since $\mathscr{H}(f \circ T)$ is symmetric, the last row of $\mathscr{H}(f \circ T)$ is zero. So $\partial/\partial x_i \partial/\partial x_n (f \circ T) = 0$ for all i. Since $\mathrm{char} K = 0$, it follows that $\partial/\partial x_n (f \circ T) \in K$. Notice that the reduced part of $f \circ T$ is just $\tilde{f} \circ T$. Consequently, $\partial/\partial x_n (\tilde{f} \circ T) = 0$ and (v) follows.

Finally assume (v). Since $\tilde{f} \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$, the last column of $\mathscr{H}(f \circ T) = \mathscr{H}(\tilde{f} \circ T)$ equals zero. So the nth standard basis vector e_n satisfies $\mathscr{H}(f \circ T) \cdot e_n = 0$. From (1), it follows that $\mathscr{H} f \cdot T e_n = 0$ and (iii) follows. □

Corollary 1.3. *$HSDP(n)$ has an affirmative answer for all $n \leqslant 4$.*

Proof. Suppose that $\mathscr{H}h$ is nilpotent. Then $\det \mathscr{H}h = 0$ in particular. By the theorem of Gordan and Nöther mentioned in the introduction, h is degenerate. If $\deg h \leqslant 1$, then the reduced part of h equals zero. If $\deg h \geqslant 2$, then h is reduced, for h is homogeneous. In either case, the reduced part of h is degenerate. Now apply Proposition 1.2, (v) $\Rightarrow$ (i). $\square$

In the remainder of this section, we assume that K is algebraically closed (in fact it is sufficient that K is closed under taking square roots).

Suppose that $\mathscr{H}h$ is nilpotent and T is orthogonal, i.e. $T^{\mathrm{t}}T = 1$. From (1), it follows that $\mathscr{H}(h \circ T)$ is nilpotent as well. So an interesting question is whether T can always be chosen orthogonal in the definition of degenerate, in which case we call *h orthogonally degenerate*. The answer is no. Take $h = (x_1 + \mathrm{i}x_2)^2$ and suppose that $h \circ T \in K[x_1]$. Then $\mathscr{H}(h \circ T)$ is of the form

$$\mathscr{H}(h \circ T) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

So $\mathscr{H}(h \circ T)$ cannot both be nilpotent and have rank 1.

We call f *isotropically degenerate* if there is an orthogonal $T \in GL_n(K)$ such that $f \circ T \in K[x_1, x_2, \ldots, x_{n-2}, x_{n-1} + \mathrm{i}x_n]$. Clearly, the above h with $n = 2$ is isotropically degenerate (take $T = 1$).

The following lemma gives a class of $f \in K[x]$ that are orthogonally degenerate.

Lemma 1.4. *Let $f \in K[x_1, x_2, \ldots, x_n]$ be reduced such that $\mathscr{H}f \cdot v = 0$ for some non-isotropic v. Then f is orthogonally degenerate.*

Proof. Replacing v by $v/\langle v, v\rangle$, we may assume that $\langle v, v\rangle^{1/2} = 1$. Then, using the Gram–Schmidt process, we can find an orthogonal $T \in GL_n(K)$ such that v is the last column of T, i.e.

$$v = T \cdot e_n. \tag{2}$$

So $\mathscr{H}f \cdot T \cdot e_n = \mathscr{H}f \cdot v = 0$. From (1), it follows that

$$\mathscr{H}(f \circ T) \cdot e_n = 0.$$

So the last column of $\mathscr{H}(f \circ T)$ equals zero. Now apply proposition 1.2, (iv) $\Rightarrow$ (v). $\square$

The following lemma, which gives a class of $f \in K[x]$ that are isotropically degenerate, is harder to prove than the above lemma. A problem is that (iv) $\Rightarrow$ (v) of Proposition 1.2 cannot be applied directly.

Lemma 1.5. *Let $f \in K[x_1, x_2, \ldots, x_n]$ be reduced such that $\mathscr{H}f \cdot v = 0$ for some isotropic $v \neq 0$. Then f is isotropically degenerate.*

Proof. Since permutation matrices are orthogonal, we may assume that $v_1 \neq 0$. Replacing v by v/v_1, we may assume that $v_1 = 1$. The first standard basis vector e_1 and the vector $\tilde{v}=-\mathrm{i}(0,v_2,v_3,\ldots,v_n)$ satisfy $\langle e_1,\tilde{v}\rangle=0$, $\langle e_1,e_1\rangle=1$, and $\langle\tilde{v},\tilde{v}\rangle=\langle e_1,e_1\rangle-\langle v,v\rangle=1$. By the Gram–Schmidt process there exists an orthogonal $T\in GL_n(K)$ such that the last two columns of T are e_1 and $\tilde{v}$, in this order. So

$$T\cdot(e_{n-1}+\mathrm{i}e_n)=(Te_{n-1}+\mathrm{i}Te_n) \tag{3}$$

$$=(e_1+\mathrm{i}\tilde{v}) \tag{4}$$

$$=v \tag{5}$$

and therefore, $\mathscr{H}f\cdot T\cdot(e_{n-1}+\mathrm{i}e_n)=\mathscr{H}f\cdot v=0$.

In order to prove this lemma, we define $S\in GL_n(K)$ by

$$S(x)=(x_1,x_2,\ldots,x_{n-2},x_{n-1}+\mathrm{i}x_n,-\mathrm{i}x_n).$$

Then $S^{-1}(x)=(x_1,x_2,\ldots,x_{n-2},x_{n-1}+x_n,\mathrm{i}x_n)$. In particular, $S^{-1}\cdot e_n=e_{n-1}+\mathrm{i}e_n$. Therefore, it follows from (3) that

$$T\cdot S^{-1}\cdot e_n=T\cdot(e_{n-1}+\mathrm{i}e_n)=v. \tag{6}$$

Consequently, $\mathscr{H}f\cdot T\cdot S^{-1}\cdot e_n=\mathscr{H}f\cdot v=0$. From (1), it follows that $g=f\circ T\circ S^{-1}$ satisfies $\mathscr{H}g\cdot e_n=0$ and therefore

$$g\in K[x_1,x_2,\ldots,x_{n-1}].$$

Consequently, $f\circ T=g\circ S$ is a polynomial in the first $n-1$ coordinates of Sx, i.e. $f\circ T\in K[x_1,x_2,\ldots,x_{n-2},x_{n-1}+\mathrm{i}x_n]$. □

The following proposition claims that a degenerate f is either orthogonally or isotropically degenerate.

Proposition 1.6. *Let $f\in K[x_1,x_2,\ldots,x_n]$ be degenerate (not necessarily reduced). Say that $f\circ T\in K[x_1,x_2,\ldots,x_{n-1}]$ with $T\in GL_n(K)$. Let $v=Te_n$ be the last column of T.*

1. *If v is not isotropic, then f is orthogonally degenerate.*
2. *If v is isotropic, then f is isotropically degenerate.*

Proof. Write $\tilde{f}$ for the reduced part of f. Suppose first that v is not isotropic. Assume without loss of generality that $\langle v,v\rangle=1$. From Lemma 1.4, it follows that there is an orthogonal $\tilde{T}\in GL_n(K)$ such that $\tilde{f}\circ\tilde{T}\in K[x_1,x_2,\ldots,x_{n-1}]$. Without loss of generality, we may assume that $f(0)=0$. Since $f-\tilde{f}$ is linear, it follows that

$$\begin{aligned}(f\circ\tilde{T})&=((f-\tilde{f})\circ\tilde{T})+(\tilde{f}\circ\tilde{T})\\&=((f-\tilde{f})\circ(\tilde{T}-T))+((f-\tilde{f})\circ T)+(\tilde{f}\circ\tilde{T})\\&=((f-\tilde{f})\circ(\tilde{T}-T))+(f\circ T)-(\tilde{f}\circ T)+(\tilde{f}\circ\tilde{T}).\end{aligned} \tag{7}$$

According to (2), we may assume that the last column of $\tilde{T}-T$ equals zero. So $\tilde{T}-T\in K[x_1,x_2,\ldots,x_{n-1}]^n$. Since $f\circ T\in K[x_1,x_2,\ldots,x_{n-1}]$ by assumption and $\tilde{f}\circ T$ is the reduced part of $f\circ T$, it follows from (7) that $f\circ\tilde{T}\in K[x_1,x_2,\ldots,x_{n-1}]$.

Suppose next that v is isotropic. Assume without loss of generality that $v_1 = 1$. From Lemma 1.5, it follows that there is an orthogonal $\tilde{T} \in GL_n(K)$ such that $\tilde{f} \circ \tilde{T} \in K[x_1, x_2, \ldots, x_{n-1} + \mathrm{i}x_n]$. Take S as in the proof of Lemma 1.5. Similar to the case that v is orthogonal, it follows from (6) that $g = f \circ \tilde{T} \circ S^{-1} \in K[x_1, x_2, \ldots, x_{n-1}]$ and therefore $f \circ \tilde{T} = g \circ S \in K[x_1, x_2, \ldots, x_{n-1} + \mathrm{i}x_n]$. $\square$

To describe the next results, it is convenient to introduce some new notation. Let $x = x_1, x_2, \ldots, x_n$, $M \in Mat_n(K[x])$, and $f \in K[x]$. Then write x_* for $x_1, x_2, \ldots, x_{n-1}$, M_* for $(M_{ij})_{1 \leqslant i,j \leqslant n-1} \in Mat_{n-1}(K[x])$, and f_* for f, viewed as polynomial in x_* over $K[x_n]$. Similarly, we define $x_{**} = x_1, x_2, \ldots, x_{n-2}$, $M_{**} = (M_{ij})_{1 \leqslant i,j \leqslant n-2}$, etc. Also, we define $\mathscr{J}_*$ $\mathscr{J}_{**}$, $\mathscr{H}_*$, and $\mathscr{H}_{**}$ in a similar way. So we have for example $\mathscr{H}_* f_* = (\mathscr{H} f)_*$.

Suppose that $f \in K[x]$ such that $\mathscr{H} f \cdot v = 0$. Take T as in Proposition 1.6 and write $g = f \circ T$. If v is not isotropic, then $g \in K[x_*]$ and therefore

$$\mathscr{H} g = \begin{pmatrix} & & & 0 \\ & \mathscr{H}_* g_* & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Consequently

$$\mathscr{H} g \text{ is nilpotent, if and only if } \mathscr{H}_* g_* \text{ is nilpotent} \tag{8}$$

in case v is not isotropic.

If v is isotropic, then $g \in K[x_{**}, x_{n-1} + \mathrm{i}x_n]$ and therefore

$$\mathscr{H} g = \begin{pmatrix} \mathscr{H}_{**} g_{**} & w & \mathrm{i}w \\ w^{\mathrm{t}} & a & \mathrm{i}a \\ \mathrm{i}w^{\mathrm{t}} & \mathrm{i}a & -a \end{pmatrix}$$

for some $w \in K[x]^{n-2}$ and $a \in K[x]$. In order to obtain the 'isotropic analogon'

$$\mathscr{H} g \text{ is nilpotent, if and only if } \mathscr{H}_{**} g_{**} \text{ is nilpotent} \tag{9}$$

of (8), we need the following lemma.

Lemma 1.7. *Let R be a commutative ring with $\mathrm{i} = \sqrt{-1}$ and $M \in Mat_n(R)$ symmetric and of the form*

$$M = \begin{pmatrix} M_{**} & w & \mathrm{i}w \\ w^{\mathrm{t}} & a & \mathrm{i}a \\ \mathrm{i}w^{\mathrm{t}} & \mathrm{i}a & -a \end{pmatrix}$$

with $w \in R^{n-2}$ and $a \in R$. Then M is nilpotent, if and only if M_{**} is nilpotent.

Proof. Suppose by induction that M^r is of the form

$$M^r = \begin{pmatrix} (M_{**})^r & u & \mathrm{i}u \\ u^{\mathrm{t}} & b & \mathrm{i}b \\ \mathrm{i}u^{\mathrm{t}} & \mathrm{i}b & -b \end{pmatrix}. \tag{10}$$

Then,

$$M^{r+1} = M \cdot M^r = \begin{pmatrix} (M_{**})^{r+1} & M_{**}u & \mathrm{i}M_{**}u \\ (M_{**}u)^{\mathrm{t}} & \langle w,u\rangle & \mathrm{i}\langle w,u\rangle \\ (\mathrm{i}M_{**}u)^{\mathrm{t}} & \mathrm{i}\langle w,u\rangle & -\langle w,u\rangle \end{pmatrix}.$$

So M^r is of the form (10) for all r.

If M is nilpotent, then M_{**} is nilpotent as well, since $(M_{**})^r$ is a submatrix of M^r for all r. Hence assume that M_{**} is nilpotent. Take r such that $(M_{**})^r = 0$. Then

$$M^r = \begin{pmatrix} \emptyset & u & \mathrm{i}u \\ u^{\mathrm{t}} & b & \mathrm{i}b \\ \mathrm{i}u^{\mathrm{t}} & \mathrm{i}b & -b \end{pmatrix}$$

and one can easily verify that $M^{2r} = (M^r)^2 = 0$. So M is nilpotent. □

2. The main result

The following result is the main theorem of this paper. Again, we assume that K is a field of characteristic zero.

Theorem 2.1. *Let $n \geqslant 1$ and suppose that $H \in K[x]^n$ such that $\mathcal{J}H$ is nilpotent and symmetric.*

1. *If SDP(p) has an affirmative answer for all $p \leqslant n$, then $x + H$ is invertible.*
2. *If H is homogeneous, SDP(p) has an affirmative answer for all $p \leqslant n-2$, and also HSDP($n-1$) and HSDP(n) have an affirmative answer, then $x + H$ is invertible.*

Proof. In case $n = 1$, $\mathcal{J}H = 0$ and therefore H is constant and $x - H$ is the inverse of $x + H$. So assume that $n \geqslant 2$. Since $\mathcal{J}H$ is symmetric, we have $\mathcal{J}H = \mathcal{H}h$ for some reduced h. If $H(0) = 0$, then $(\mathcal{J}h)^{\mathrm{t}} = H$ as well. Since translations are invertible, we assume that $H(0) = 0$.

We shall show the following assertions.

(i) If h is degenerate and (homogeneous and H)SDP($n-1$) has an affirmative answer, then h is isotropically degenerate.

(ii) If $h \in K[x_{**}, x_{n-1} + \mathrm{i}x_n]$, then $x + H$ is invertible over K, if and only if $x_{**} + H_{**}$ is invertible over $K(x_{n-1} + \mathrm{i}x_n)$.

Suppose first that these assertions hold. Since $SDP(n)$ resp. $HSDP(n)$ is assumed to have an affirmative answer, it follows from (i) and Proposition 1.2 that h is isotropically degenerate. Suppose that conclusion 1. resp 2. of this theorem does not hold. Take n minimal such that H satisfies the conditions of this theorem, but $x + H$ is not invertible. Since h is isotropically degenerate, there is an orthogonal $T \in GL_n(K)$ such that $h \circ T \in K[x_{**}, x_{n-1} + \mathrm{i}x_n]$.

If $n \geqslant 3$, then it follows from (9) and (ii) that there is a $G \in K(y)[x_{**}]^n$ such that $\mathcal{J}G$ is nilpotent and symmetric, but $x_{**} + G$ is not invertible over $K(y)$. Since G satisfies the conditions of this theorem as well as H, we have a contradiction, so $n \leqslant 2$. The case $n = 1$ is trivial, so assume that $n = 2$. Take $G = (H_1, H_2, 0)$. Then $x + H$ is invertible, if and only if $(x, x_3) + G$ is invertible. Furthermore, the invertibility of G reduces to the case $n = 1$ of this theorem, which is trivially satisfied.

First, we show assertion (i). Suppose that h is (homogeneous and) not isotropically degenerate. Since h is assumed to be degenerate, h is orthogonally degenerate according to Proposition 1.6. Take T orthogonal such that $h \circ T \in K[x_*]$. Since $g = h \circ T$ is reduced, it follows from (8) and the fact that $(H)SDP(n-1)$ has an affirmative answer that g is degenerate, so g is either isotropically or orthogonally degenerate according to Proposition 1.6. If g is isotropically degenerate, then h is isotropically degenerate as well. If, on the other hand, g is orthogonally degenerate, then there is an orthogonal $S \in GL_n(K)$ such that $g \circ S \in K[x_{**}]$. Therefore $h \circ (T \circ S) \in K[x_{**}] \subseteq K[x_{**}, x_{n-1} + \mathrm{i}x_n]$ and h is isotropically degenerate.

Next, we show assertion (ii). Suppose that $h \in K[x_{**}, x_{n-1} + \mathrm{i}x_n]$, say that $h = g(x_{**}, x_{n-1} + \mathrm{i}x_n)$ with $g \in K[x_*]$. Put $H_{**} = (J_{**}h)^{\mathrm{t}}$. Then

$$\begin{aligned} P_1 &:= x + H \\ &= \left(x_{**} + H_{**}, x_{n-1} + \left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1} + \mathrm{i}x_n),\right. \\ &\qquad \left. x_n + \mathrm{i}\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1} + \mathrm{i}x_n)\right) \end{aligned}$$

is invertible, if and only if

$$\begin{aligned} P_2 &:= (x_{**}, x_{n-1} + \mathrm{i}x_n, x_n) \circ P_1 \\ &= \left(x_{**} + H_{**}, x_{n-1} + \mathrm{i}x_n, x_n + \mathrm{i}\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1} + \mathrm{i}x_n)\right) \end{aligned}$$

is invertible. Put $G_{**} = \mathcal{J}_{**}g$, then $G_{**} = H_{**}(x_{**}, x_{n-1} - \mathrm{i}x_n, x_n)$. So P_2 is invertible, if and only if

$$\begin{aligned} P_3 &:= P_2 \circ (x_{**}, x_{n-1} - \mathrm{i}x_n, x_n) \\ &= \left(x_{**} + G_{**}, x_{n-1}, x_n + \mathrm{i}\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_{**}, x_{n-1})\right) \end{aligned}$$

is invertible. P_3 is invertible, if and only

$$P_4 := \left(x_*, x_n - \mathrm{i}\left(\frac{\partial}{\partial x_{n-1}} g\right)(x_*)\right) \circ P_3 = (x_{**} + G_{**}, x_{n-1}, x_n)$$

is invertible. Since $G_{**} \in K[x_*]^{n-2}$, it follows that P_4 is invertible, if and only if $x_{**} + G_{**}$ is invertible over $K[x_{n-1}]$, i.e. if and only if $x_{**} + H_{**}$ is invertible over $K[x_{n-1} + \mathrm{i}x_n]$. By [2, Lemma 1.1.8], this last statement is equivalent to the assertion that $x_{**} + H_{**}$ is invertible over $K(x_{n-1} + \mathrm{i}x_n)$. This gives assertion (ii). $\square$

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